Exercise 4.3

Wouter

VU

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Exercise 4.3

Exercise 4.3 (Testing against Hölder alternatives, pointwise separation). Let $\beta > 0$. Let $C_1^\beta[0,1]$ be the unit ball of the Hölder space $C^\beta[0,1]$.

(i) Show that in the white noise model, there exists no consistent test for the hypotheses $H_0 : f = 0$ against $H_1 : f \in C_1^\beta[0,1] \backslash \{0\}$.

(ii) Let $t_0 \in (0,1)$. Show that if r_n is of smaller order than $n^{-\beta/(1+2\beta)}$, then there exists no consistent test for the hypotheses $H_0 : f = 0$ against $H_1 : f \in \{f \in C_1^\beta[0,1] : |f(t_0)| \geq r_n\}$.

(iii) Let $t_0 \in (0,1)$. Show that if r_n is of larger order than $n^{-\beta/(1+2\beta)}$, then there does exist a consistent test for the hypotheses $H_0 : f = 0$ against $H_1 : f \in \{f \in C_1^\beta[0,1] : |f(t_0)| \geq r_n\}$. (Hint: use the estimator of Example 3.4.3 as test statistic and use the result (3.12) to bound the errors of the test.)

Holder norm

Concretely, for $f \in C^\beta[0,1]$, we define the Hölder norm of order β by

$$\|f\|_{C^\beta} = \max_{k \leq \underline{\beta}} \|f^{(k)}\|_\infty + \sup_{s \neq t} \frac{|f^{(\underline{\beta})}(t) - f^{(\underline{\beta})}(s)|}{|t-s|^{(\beta - \underline{\beta})}}.$$

The Hölder ball of order β and radius R is then defined as $C_R^\beta[0,1] = \{f \in C^\beta[0,1] : \|f\|_{C^\beta} \leq R\}$. Inspection of Example 3.4.3 shows that for the appropriate choice of the bandwidth h, the kernel estimator $\hat{f}_h$ satisfies

$$\sup_{f \in C_1^\beta[0,1]} \mathrm{E}_f(\hat{f}_h(t) - f(t))^2 \leq \text{const.} \times n^{-2\beta/(1+2\beta)} \tag{3.12}$$

(i) and (ii)

Example 4.2.4 (Detecting a smooth bump at a fixed point). Consider the function

$$f_\sigma(x) = \sigma^\beta K\Big(\frac{x - 1/2}{\sigma}\Big), \qquad x \in [0, 1],$$

where K is C^∞ and has compact support, and $\beta, \sigma > 0$. This function has C^β-norm of the order 1 (i.e. the supremum of the C^β-norm over σ is bounded) and its squared L^2-norm of the order $\sigma^{1+2\beta}$ (see Exercise 4.2). So the spike is consistently detectable if and only if $n\sigma^{1+2\beta} \to \infty$, that is, if and only the height $f_\sigma(1/2)$ of the bump is of of larger order than $n^{-\beta/(1+2\beta)}$. ■

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Take

$$f_n(x) = \frac{1}{B} f_{(Br_n/K(0))^{1/\beta}}(x + 1/2 - t_0)$$

$$f_n(t_0) = r_n$$

$$n\sigma^{1+2\beta} = n(Br_n/K(0))^{(1+2\beta)/\beta} = no(n^{-1}) = o(1)$$

L2 norm f

$$||f_\sigma||_2^2 = \int_0^1 \sigma^{2\beta} K((x-1/2)/\sigma)dx = \sigma^{2\beta+1} \int_{-1/2\sigma}^{1/2\sigma} K(y)dy \sim \sigma^{2\beta+1}$$

(iii)

$$\hat{f}_h(t) = \int_0^1 \frac{1}{h} K\left(\frac{t-s}{h}\right) dX_s, \qquad t \in [0,1],$$

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$$\sup_{f \in C_1^\beta[0,1]} \mathrm{E}_f(\hat{f}_h(t) - f(t))^2 \leq \text{const.} \times n^{-2\beta/(1+2\beta)} \qquad (3.12)$$

$$E_0 \hat{f}_h^2(t_0) = O(n^{-2\beta/(1+2\beta)}) = o(r_n^2)$$

$$E_f((\hat{f}_h(t_0) - f(t_0))^2) = O(n^{-2\beta/(1+2\beta)}) = o(r_n^2)$$

We can consistently separate $H_0 : f = 0$ from $H_1 : f \in C_1^\beta[0,1]\backslash\{|f(t_0)| < r_n\}$ by taking $\phi = 1(|\hat{f}_h(t_0)| > r_n/2)$

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$$P_0(|\hat{f}_h(t_0)| > r_n/2) = P_0(\hat{f}_h^2(t_0) > r_n^2/4) \leq E(\hat{f}_h^2(t_0))/(r_n^2/4) = o(1)$$

$$P_f(|\hat{f}_h(t_0) - f(t_0)| > r_n/2) = P_0((\hat{f}_h(t_0) - f(t_0))^2 > r_n^2/4) \leq$$

$$E((\hat{f}_h(t_0) - f(t_0))^2)/(r_n^2/4) = o(1)$$