Exercise 4.3

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Exercise 4.3 (Testing against Hölder alternatives, pointwise separation). Let $\beta > 0$. Let $C_1^\beta [0, 1]$ be the unit ball of the Hölder space $C^\beta [0, 1]$.

(i) Show that in the white noise model, there exists no consistent test for the hypotheses $H_0 : f = 0$ against $H_1 : f \in C_1^\beta [0, 1] \setminus \{0\}$.

(ii) Let $t_0 \in (0, 1)$. Show that if $r_n$ is of smaller order than $n^{-\beta/(1+2\beta)}$, then there exists no consistent test for the hypotheses $H_0 : f = 0$ against $H_1 : f \in \{f \in C_1^\beta [0, 1] : |f(t_0)| \geq r_n\}$.

(iii) Let $t_0 \in (0, 1)$. Show that if $r_n$ is of larger order than $n^{-\beta/(1+2\beta)}$, then there exists a consistent test for the hypotheses $H_0 : f = 0$ against $H_1 : f \in \{f \in C_1^\beta [0, 1] : |f(t_0)| \geq r_n\}$. (Hint: use the estimator of Example 3.4.3 as test statistic and use the result (3.12) to bound the errors of the test.)
Concretely, for \( f \in C^\beta[0,1] \), we define the Hölder norm of order \( \beta \) by

\[
\|f\|_{C^\beta} = \max_{k \leq \beta} \|f^{(k)}\|_\infty + \sup_{s \neq t} \frac{|f^{(\beta)}(t) - f^{(\beta)}(s)|}{|t - s|^{(\beta - \beta)}}.
\]

The Hölder ball of order \( \beta \) and radius \( R \) is then defined as \( C^\beta_R[0,1] = \{ f \in C^\beta[0,1] : \|f\|_{C^\beta} \leq R \} \). Inspection of Example 3.4.3 shows that for the appropriate choice of the bandwidth \( h \), the kernel estimator \( \hat{f}_h \) satisfies

\[
\sup_{f \in C^\beta_1[0,1]} E_f (\hat{f}_h(t) - f(t))^2 \leq \text{const.} \times n^{-2\beta/(1+2\beta)}
\]

(3.12)
Example 4.2.4 (Detecting a smooth bump at a fixed point). Consider the function
\[ f_\sigma(x) = \sigma^\beta K\left(\frac{x - 1/2}{\sigma}\right), \quad x \in [0, 1], \]
where \( K \) is \( C^\infty \) and has compact support, and \( \beta, \sigma > 0 \). This function has \( C^\beta \)-norm of the order 1 (i.e. the supremum of the \( C^\beta \)-norm over \( \sigma \) is bounded) and its squared \( L^2 \)-norm of the order \( \sigma^{1+2\beta} \) (see Exercise 4.2). So the spike is consistently detectable if and only if \( n\sigma^{1+2\beta} \to \infty \), that is, if and only the height \( f_\sigma(1/2) \) of the bump is of of larger order than \( n^{-\beta/(1+2\beta)} \). \( \blacksquare \)
(i) and (ii)

Example 4.2.4 (Detecting a smooth bump at a fixed point). Consider the function

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Take

\[ f_n(x) = \frac{1}{B} f_{(Br_n/K(0))^{1/\beta}}(x + 1/2 - t_0) \]

\[ f_n(t_0) = r_n \]

\[ n\sigma^{1+2\beta} = n(Br_n/K(0))^{(1+2\beta)/\beta} = no(n^{-1}) = o(1) \]
\[ \|f_\sigma\|_2^2 = \int_0^1 \sigma^{2\beta} K((x-1/2)/\sigma) dx = \sigma^{2\beta+1} \int_{-1/2\sigma}^{1/2\sigma} K(y) dy \sim \sigma^{2\beta+1} \]
(iii)

\[ \hat{f}_h(t) = \int_0^1 \frac{1}{h} K\left(\frac{t-s}{h}\right) dX_s, \quad t \in [0, 1], \]
\[ \hat{f}_h(t) = \int_0^1 \frac{1}{h} K\left(\frac{t-s}{h}\right) dX_s, \quad t \in [0, 1], \]

Concretely, for \( f \in C^\beta[0, 1] \), we define the Hölder norm of order \( \beta \) by

\[ \|f\|_{C^\beta} = \max_{k \leq \beta} \|f^{(k)}\|_\infty + \sup_{s \neq t} \frac{|f^{(\beta)}(t) - f^{(\beta)}(s)|}{|t-s|^{(\beta-\beta)}}. \]

The Hölder ball of order \( \beta \) and radius \( R \) is then defined as \( C^\beta_R[0, 1] = \{f \in C^\beta[0, 1] : \|f\|_{C^\beta} \leq R\} \). Inspection of Example 3.4.3 shows that for the appropriate choice of the bandwidth \( h \), the kernel estimator \( \hat{f}_h \) satisfies

\[ \sup_{f \in C^\beta_1[0,1]} \mathbb{E}_f (\hat{f}_h(t) - f(t))^2 \leq \text{const.} \times n^{-2\beta/(1+2\beta)} \quad (3.12) \]
\[ E_0 \hat{f}^2_h(t_0) = O(n^{-2\beta/(1+2\beta)}) = o(r_n^2) \]
\[ E_f((\hat{f}_h(t_0) - f(t_0))^2) = O(n^{-2\beta/(1+2\beta)}) = o(r_n^2) \]

We can consistently separate \( H_0 : f = 0 \) from \( H_1 : f \in C_1^\beta [0,1] \setminus \{ |f(t_0)| < r_n \} \) by taking \( \phi = 1(|\hat{f}_h(t_0)| > r_n/2) \).
\begin{align*}
E_0 \hat{f}_h^2(t_0) &= O(n^{-2\beta/(1+2\beta)}) = o(r_n^2) \\
E_f((\hat{f}_h(t_0) - f(t_0))^2) &= O(n^{-2\beta/(1+2\beta)}) = o(r_n^2)
\end{align*}

We can consistently separate $H_0 : f = 0$ from $H_1 : f \in C_1^\beta[0,1] \setminus \{|f(t_0)| < r_n\}$ by taking $\phi = 1(|\hat{f}_h(t_0)| > r_n/2)$

\begin{align*}
P_0(|\hat{f}_h(t_0)| > r_n/2) &= P_0(\hat{f}_h^2(t_0) > r_n^2/4) \leq E(\hat{f}_h^2(t_0))/(r_n^2/4) = o(1) \\
P_f(|\hat{f}_h(t_0) - f(t_0)| > r_n/2) &= P_0((\hat{f}_h(t_0) - f(t_0))^2 > r_n^2/4) \leq \\
E((\hat{f}_h(t_0) - f(t_0))^2)/(r_n^2/4) &= o(1)
\end{align*}