Exercise 4.5

Exercise 4.5 (Testing against Hölder alternatives, uniform separation). Let $\beta > 0$. Let $C_1^\beta[0,1]$ be the unit ball of the Hölder space $C^\beta[0,1]$.

(i) Show that if r_n is of smaller order than $(n/\log n)^{-\beta/(1+2\beta)}$, then there exists no consistent test for the hypotheses $H_0 : f = 0$ against $H_1 : f \in \{f \in C_1^\beta[0,1] : \sup_{u<t<v} |f(t)| \geq r_n\}$.

(ii) Show that if r_n is of larger order than $(n/\log n)^{-\beta/(1+2\beta)}$, then there does exist a consistent test for the hypotheses $H_0 : f = 0$ against $H_1 : f \in \{f \in C_1^\beta[0,1] : \sup_{u<t<v} |f(t)| \geq r_n\}$. (Hint: use the estimator of Example 3.4.4 as test statistic and use the result (3.13) to bound the errors of the test.)

Explain the differences with the results of Exercise 4.3.

(i) r_n is of smaller order than $(\frac{n}{\log n})^{-\beta/(1+2\beta)}$

Example 4.3.3 (Detecting a smooth bump at one of many locations). Fix $0 < u < v < 1$. For $x_1, \ldots, x_k \in (u, v)$, consider the bump functions $f_{\sigma,i}$ defined by

$$f_{\sigma,i}(x) = \varepsilon\sigma^\beta K\Big(\frac{x - x_i}{\sigma}\Big), \qquad x \in [0,1],$$

where K is a C^∞ function with compact support and $\beta, \sigma, \varepsilon > 0$. As in Example 4.2.4, the C^β-norm of these functions is uniformly bounded by 1 if ε is small enough and their squared L^2-norm is bounded by $C\varepsilon^2\sigma^{1+2\beta}$ for some $C > 0$.

- Position $x_1, \ldots, x_k \in (u, v)$
- No consistent test for $H_0 : f_0 = 0$ against $H_1 : f \in \{f_{\sigma,1}, \ldots, f_{\sigma,k}\}$ if $\sigma e^{nC\epsilon^2\sigma^{1+2\beta}} \leq \delta$ for some $\delta \in (0, 1)$.
- This condition is fulfilled for ϵ small enough and $\sigma = (\frac{n}{\log n})^{-\frac{1}{1+2\beta}}$.

Consider $f_{\sigma,i} = \epsilon\sigma^{\beta} K\left(\frac{x-x_i}{\sigma}\right)$, where $\sigma = (\frac{r_n}{\epsilon K(0)})^{1/\beta}$, $i = 1, \ldots, k$, $k \sim \frac{1}{\sigma}$:

(i) $f_{\sigma,i} \in C_1^{\beta}[0,1]$ for all i if ϵ small enough

(ii) $r_n = o\left((\frac{n}{\log n})^{-\beta/(1+2\beta)}\right) \Rightarrow \sigma = o\left((\frac{n}{\log n})^{-1/(1+2\beta)}\right)$

(iii) $\sup_t |f_{\sigma,i}(t)| = \epsilon\sigma^{\beta} K(0) = r_n$

(ii) r_n is of larger order than $(\frac{n}{\log n})^{-\beta/(1+2\beta)}$

In Ex.3.4.4, define

$$\hat{f}_h(t) = \int_0^1 \frac{1}{h} K\left(\frac{t-s}{h}\right) dX_s, \qquad t \in [0,1]$$

The uniform risk

$$\mathbb{E}_f \sup_t |\hat{f}_h(t) - f(t)| = O\left((\frac{n}{\log n})^{-\beta/(1+2\beta)} \right) = o(r_n)$$

Consider $\phi = \mathbb{1}_{\{\sup_t |\hat{f}_h| \geq r_n/2\}}$

$$\begin{aligned}
\mathbb{E}_0 \phi =& \mathbb{P}_0(\sup_t |\hat{f}_h| \geq r_n/2) \leq \frac{\mathbb{E}_0 \sup_t |\hat{f}_h|}{r_n/2} = o(1) \\
\sup_{f: \sup |f| \geq r_n} & \mathbb{E}_f(1-\phi) = \sup_f \mathbb{P}_f(\sup_t |\hat{f}_h| < r_n/2) \\
& \leq \sup_f \mathbb{P}_f(\sup_t |\hat{f}_h - f| \geq r_n/2) \leq \sup_f \frac{\mathbb{E}_f \sup |\hat{f}_h - f|}{r_n/2} = o(1)
\end{aligned}$$

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[1] $\sup_t |\hat{f}_h - f| \geq \sup_t(|f| - |\hat{f}_h|) \geq r_n - r_n/2 = r_n/2$