Ex 4.1. Complete the proof for the following:
For all $f_1, f_2 \in L^2[0,1]$,

$$\frac{1}{2}\inf_{\phi}\left(\mathbb{E}_{f_0}\phi + \mathbb{E}_{f_1}(1-\phi)\right) = 1 - \Phi\left(\frac{1}{2}\sqrt{n}||f_1 - f_0||_2\right)$$

where the infimum is over all tests ϕ.

Test $H_1: \ f = f_1$ vs $H_0: \ f = f_0$

Minimax risk:

$$\inf_{\phi} \left(\mathbb{E}_{f_0}\phi + \sup_{f \in \mathcal{F}} \mathbb{E}_f (1-\phi) \right) = \inf_{\phi} \left(\mathbb{E}_{f_0}\phi + \mathbb{E}_{f_1}(1-\phi) \right)$$

$$= \inf_{\alpha \in (0,1)} \left(\alpha + \inf_{\phi \text{ of level } \alpha} \mathbb{E}_{f_1}(1-\phi) \right)$$

$$= \inf_{\alpha \in (0,1)} \left(\alpha + 1 - \Phi\left(\sqrt{n}||f_1 - f_0||_2 - \xi_{1-\alpha} \right) \right)$$

Goal:

$$\frac{1}{2} \inf_{\phi} \left(\mathbb{E}_{f_0}\phi + \mathbb{E}_{f_1}(1-\phi) \right) = 1 - \Phi\left(\frac{1}{2}\sqrt{n}||f_1 - f_0||_2 \right).$$

Method: differentiate $\alpha \mapsto \left(\alpha + 1 - \Phi\left(\sqrt{n}||f_1 - f_0||_2 - \xi_{1-\alpha} \right) \right)$:

$$0 = 1 + \psi\left(\sqrt{n}||f_1 - f_0||_2 - \xi_{1-\alpha} \right) \cdot \frac{\mathrm{d}\xi_{1-\alpha}}{\mathrm{d}\alpha}$$

(ψ: standard normal density)

What is $\frac{\mathrm{d}}{\mathrm{d}\alpha}\xi_{1-\alpha}$?
$1-\alpha = \Phi\left(\xi_{1-\alpha}\right) \leftrightarrow \Phi^{-1}\left(1-\alpha\right) = \xi_{1-\alpha}$
Differentiate $\Phi^{-1}\left(\Phi(\xi_{1-\alpha})\right) = \xi_{1-\alpha}$ w.r.t $\xi_{1-\alpha}$ w.r.t. $\xi_{1-\alpha}$:
$(\Phi^{-1})'\left(\Phi(\xi_{1-\alpha})\right)\psi(\xi_{1-\alpha}) = 1$, so

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\xi_{1-\alpha} = -(\Phi^{-1})'\left(1-\alpha\right) = -1/\psi(\xi_{1-\alpha}).$$

$\rightarrow$ Can plug this into

$$0 = 1 + \psi\left(\sqrt{n}||f_1 - f_0||_2 - \xi_{1-\alpha}\right) \cdot \frac{\mathrm{d}\xi_{1-\alpha}}{\mathrm{d}\alpha}$$

to find α_{min}.

$0 = 1 - \psi\left(\sqrt{n}||f_1 - f_0||_2 + \xi_{1-\alpha}\right) \cdot 1/\psi(\xi_{1-\alpha}).$
$\Rightarrow \psi(\xi_{1-\alpha}) = \psi\left(\sqrt{n}||f_1 - f_0||_2 - \xi_{1-\alpha}\right)$
$\Rightarrow \pm\xi_{1-\alpha} = \sqrt{n}||f_1 - f_0||_2 - \xi_{1-\alpha}.$
Assume $f_0 \neq f_1 \Rightarrow \xi_{1-\alpha} = \frac{1}{2}\sqrt{n}||f_1 - f_0||_2$
Then α_{min} is s.t. $\xi_{1-\alpha} = \frac{1}{2}\sqrt{n}||f_1 - f_0||_2$, i.e.
$\alpha_{min} = 1 - \Phi\left(\frac{1}{2}\sqrt{n}||f_1 - f_0||_2\right).$
Concluding:

$$\inf_{\alpha\in(0,1)} \left(\alpha + 1 - \Phi\left(\sqrt{n}||f_1 - f_0||_2 - \xi_{1-\alpha}\right)\right) = 2 - 2\Phi\left(\frac{1}{2}\sqrt{n}||f_1 - f_0||_2\right)$$

Hence:

$$\frac{1}{2}\inf_{\phi}\left(\mathbb{E}_{f_0}\phi + \mathbb{E}_{f_1}(1-\phi)\right) = 1 - \Phi\left(\frac{1}{2}\sqrt{n}||f_1 - f_0||_2\right) \qquad .$$