Statistical theory for high and infinite-dimensional models

Exercise 3.2

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Problem statement

Let $\underline{Y} \sim N_n(\underline{\theta}, \mathbf{I})$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be an almost everywhere differentiable function such that $E_\theta|(\partial f/\partial x_i)(\underline{Y})| < \infty$ for $i = 1, ..., n$. Then for $i = 1, ..., n$,

$$E_\theta(Y_i - \theta_i)f(\underline{Y}) = E_\theta \frac{\partial f}{\partial x_i}(\underline{Y})$$

Some results

If $Y_i \sim N(\theta_i, 1)$ for $i = 1, ..., n$ with $p(y_i) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}(y_i - \theta_i)^2)$ then

$$p'(y_i) = (\theta_i - y_i)p(y_i)$$

and by the fundamental theorem of calculus we have that

$$p(y_i) = \int_{-\infty}^{y_i} (\theta_i - t)p(t)dt = \int_{y_i}^{\infty} (t - \theta_i)p(t)dt$$

Some more results

Fubini's theorem states that we may switch the order of integration if the double integral yields a finite answer when the integrand is replaced by it's absolute value. Hence,

$$\int_X (\int_Y f(x,y)dy)dx = \int_Y (\int_X f(x,y)dx)dy$$

Solution part 1

Define f to instead be of the form $f(\cdot, \underline{y}_{-i}) : \mathbb{R} \to \mathbb{R}$ for each $i = 1, ..., n$ and almost every $\underline{y}_{-i} \in \mathbb{R}^{n-1}$. Hence in this case f is a function of it's i^{th} argument. This can be done due to the fact that f is almost everywhere differentiable. We then proceed by firstly calculating $E[(\partial f / \partial x_i)(\underline{Y}) | \underline{Y}_{-i}]$ and then subsequently taking the expectation over all $\underline{Y}_{-i}$ to get the result.

Solution part 2

$$E[(\partial f/\partial x_i)(\underline{Y})|\underline{Y}_{-i}]$$

$$= \int_{-\infty}^{\infty} \frac{\partial f}{\partial x_i}(y_i, \underline{Y}_{-i})p(y_i)dy_i$$

$$= \int_{0}^{\infty} \frac{\partial f}{\partial x_i}(y_i, \underline{Y}_{-i})p(y_i)dy_i + \int_{-\infty}^{0} \frac{\partial f}{\partial x_i}(y_i, \underline{Y}_{-i})p(y_i)dy_i$$

$$= \int_{0}^{\infty} \frac{\partial f}{\partial x_i}(y_i, \underline{Y}_{-i}) \left\{ \int_{y_i}^{\infty}(t-\theta_i)p(t)dt \right\} dy_i$$
$$- \int_{-\infty}^{0} \frac{\partial f}{\partial x_i}(y_i, \underline{Y}_{-i}) \left\{ \int_{-\infty}^{y_i}(t-\theta_i)p(t)dt \right\} dy_i$$

$$= \int_{0}^{\infty}(t-\theta_i)p(t) \left\{ \int_{0}^{t} \frac{\partial f}{\partial x_i}(y_i, \underline{Y}_{-i})dy_i \right\} dt$$
$$- \int_{-\infty}^{0}(t-\theta_i)p(t) \left\{ \int_{t}^{0} \frac{\partial f}{\partial x_i}(y_i, \underline{Y}_{-i})dy_i \right\} dt$$

Solution part 3

$$= \int_0^{\infty} (t-\theta_i) p(t) [f(t, \underline{Y}_{-i}) - f(0, \underline{Y}_{-i})] dt$$
$$- \int_{-\infty}^{0} (t-\theta_i) p(t) [f(0, \underline{Y}_{-i}) - f(t, \underline{Y}_{-i})]$$

$$= \int_{-\infty}^{\infty} (t-\theta_i) [f(t, \underline{Y}_{-i}) - f(0, \underline{Y}_{-i})] p(t) dt$$

$$= E[(Y_i - \theta_i) \left\{ f(\underline{Y}_i, \underline{Y}_{-i}) - f(0, \underline{Y}_{-i}) \right\} | \underline{Y}_{-i}]$$

$$= E[(Y_i - \theta_i) f(\underline{Y}_i, \underline{Y}_{-i}) | \underline{Y}_{-i}]$$

Solution part 4

As a final step we take expectation over all $\underline{Y}_{-i}$ to get the result that for $i = 1, ..., n$

$$E_\theta (Y_i - \theta_i) f(Y) = E_\theta \frac{\partial f}{\partial x_i}(Y)$$

Thank you for your time. Any questions?