1 Local equivalence of period drift estimation and signal-in-white-noise

Suppose we observe a process $X = (X_t : t \in [0, T])$ satisfying the SDE

$$\mathrm{d}\, X_t = b(X_t)\,\mathrm{d}\, t + \mathrm{d}\, W_t$$

where b belongs to the space of C_o^1 of 1 periodic, continuously differentiable functions. The statistical goal is to estimate the function b. For such a process we have a law of large numbers which asserts that if $F \in C_o^1$, then as $T \to \infty$

$$\frac{1}{T}\int_0^T f(X_t)\,\mathrm{d}\, t \to \int_0^1 f(x)\rho(x)\,\mathrm{d}\, x,$$

almost surely, where ρ is the probability density on $[0, 1]$ given by

$$\rho(x) = Ce^{2\int_0^x b(y)}\,\mathrm{d}\, y$$

where C is the appropriate normalizing constant. Moreover, we have a central limit theorem which asserts that for $f \in C_o^1$,

$$\frac{1}{\sqrt{T}}\int_0^T f(X_t)\,\mathrm{d}\, W_t \overset{\mathrm{d}}{\to} N\left(0, \int_0^1 f^2(x)\rho(x)\,\mathrm{d}\, x\right)$$

as $T \to \infty$.

Let $P_{b,T}$ be the law that the process X induces on the space $C[0, T]$ of continuous functions on $[0, T]$. By a slight extension of theorem 2.4.1, the likelihood ratio $\frac{\mathrm{d}\, P_{b,T}}{\mathrm{d}\, P_{b_0,T}}$ for two functions $b, b_0 \in C_o^1$ is given by

$$\frac{\mathrm{d}\, P_{b,T}}{\mathrm{d}\, P_{b_0,T}} = \exp\left(\int_0^T (b - b_0)(X_t)\,\mathrm{d}\, W_t - \frac{1}{2}\int_0^T (b - b_0)^2(X_t)\,\mathrm{d}\, t\right),$$

where W is a $P_{b_0,T}$ Brownian motion. Use this in combination with the law of large numbers and central limit theorem given above to convince yourself that if we know in advance that b is close to another function $b_0 \in C_o^1$, then for $T \to \infty$, the statistical problem under consideration is close to a signal in white noise problem. Which one exactly?

2 solution Sketch

First we observe that

$$\frac{1}{2}\int_0^T (b - b_0)^2(X_t)\,\mathrm{d}\, t = \frac{1}{2}\frac{T}{T}\int_0^T (b - b_0)^2(X_t)\,\mathrm{d}\, t,$$

and we know a limit for

$$\frac{1}{T}\int_0^T (b-b_0)^2(X_t)\,\mathrm{d}\,t$$

This almost surely is

$$\int_0^1 (b(x)-b_0(x))\rho(x)\,\mathrm{d}\,x$$

So for finite but large enough T we know that

$$\frac{1}{T}\int_0^T (b-b_0)^2(X_t)\,\mathrm{d}\,t \approx \int_0^1 (b(x)-b_0(x))^2\rho(x)\,\mathrm{d}\,x$$

If we pick f the function $x \mapsto b(x)\sqrt{\rho(x)}$ and f_0 being $x \mapsto b_0(x)\sqrt{\rho(x)}$. Then

$$||f-f_0||_2^2 = \int_0^1 (f-f_0)^2(x)\,\mathrm{d}\,x = int_0^1(b(x)-b_0(x))^2\rho(x)\,\mathrm{d}\,x$$

Now look at the Ito integral, we again can rewrite this into some form we are interested in. We now multiply and divide by $\sqrt{T}$, this yields

$$\frac{\sqrt{T}}{\sqrt{T}}\int_0^T (b-b_0)(X_t)\,\mathrm{d}\,W_t.$$

Now we know a limiting distribution for

$$\frac{1}{\sqrt{T}}\int_0^T (b-b_0)(X_t)\,\mathrm{d}\,W_t,$$

namely this is normally distributed with mean zero and variance $\int_0^1 (b-b_0)^2(x)\rho(x)\,\mathrm{d}\,x$.

Now take a look at

$$\int_0^1 (f-f_0)(t)\,\mathrm{d}\,W_t.$$

This is an Ito integral with a deterministic function. For this we have a theorem which tells us the distribution of the integral. If we immediately specialize to this case we get that our integral is a normal random variable with mean zero and variance

$$\int_0^1 (f-f_0)^2(t)\,\mathrm{d}\,t.$$

This is exactly $\int_0^1 (b(x)-b_0(x))^2\rho(x)\,\mathrm{d}\,x$, the limiting distribution of

$$\frac{1}{\sqrt{T}}\int_0^T (b-b_0)(X_t)\,\mathrm{d}\,W_t$$

For finite but large enough T we thus get

$$\frac{1}{\sqrt{T}}\int_0^T (b-b_0)(X_t)\,\mathrm{d}\,W_t \approx \int_0^1 (f-f_0)(t)\,\mathrm{d}\,W_t.$$

Now we can summarize:

For $f(x) = b(x)\sqrt{\rho(x)}, f_0(x) = b_0(x)\sqrt{\rho(x)}$ we have the following two approximations:

$$\frac{1}{T}\int_0^T (b-b_0)^2(X_t)\,\mathrm{d}\,t \approx ||f-f_0||_2^2$$

$$\frac{1}{\sqrt{T}}\int_0^T (b-b_0)(X_t)\,\mathrm{d}\,W_t \approx \int_0^1 (f-f_0)(t)\,\mathrm{d}\,W_t.$$

So if you are sloppy, you might want to say

$$\exp\left(\int_0^T (b-b_0)(X_t)\,\mathrm{d}\,W_t - \frac{1}{2}\int_0^T (b-b_0)^2(X_t)\,\mathrm{d}\,t\right)$$

is about equal to

$$\exp\left(\sqrt{T}\int_0^1 (f-f_0)(t)\,\mathrm{d}\,W_t - \frac{1}{2}T||f-f_0||_2^2\right)$$

However, we are multiplying by T, which blows up the errors. For this we need to control the convergence rate. This is however outside the scope of this exercise, and the above discussion yields the system we want. It is intended that we ignore the blowing up of errors, and just act that it works even if we multiply by T or $\sqrt{T}$.

2.1 The final system

We pick $n = T$ and f, f_0 the functions

$$f(x) = b(x)\sqrt{\rho(x)}, f_0(x) = b_0(x)\sqrt{\rho(x)}$$

Note that ρ here is the corresponding density with respect to b_0. Then the the period drift estimation for b, b_0 and signal-in-white-noise f, f_0 are approximately the same in the sense that their likelihood functions becomes closer:

$$\exp\left(\int_0^T (b-b_0)(X_t)\,\mathrm{d}\,W_t - \frac{1}{2}\int_0^T (b-b_0)^2(X_t)\,\mathrm{d}\,t\right)$$

is almost the same as

$$\exp\left(\sqrt{T}\int_0^1 (f-f_0)(t)\,\mathrm{d}\,W_t - \frac{1}{2}T||f-f_0||_2^2\right)$$

For large T.