1 Admissibility of the MLE in the normal mean model for $n = 1$

We assume $Y \sim N(\theta, 1)$. THe goal is to prove that there is no other estimator $\hat{\theta}$ such that $\mathbb{E}_\theta(\hat{\theta} - \theta)^2 \leq \mathbb{E}_\theta(Y - \theta)^2$ for all $\theta \in \mathbb{R}$, with strict inequality for some $\theta \in \mathbb{R}$.

For $\tau > 0$, consider the $N(0, \tau)$ prior on the parameter θ. Denote the corresponding density by π_τ.

1.1 First part

Show that if an estimator $\hat{\theta}$ as described above would exists, then there would exists an $\epsilon > 0$ and $\theta_0 < \theta_1$ such that

$$1 - \int \mathbb{E}_\theta(\hat{\theta} - \theta)^2 \pi_\tau(\theta) \,\mathrm{d}\,\theta \geq \epsilon \int_{\theta_0}^{\theta_1} \pi_\tau(\theta) \,\mathrm{d}\,\theta.$$

As a first step, we are going to show that for any $\hat{\theta}$ the map $\theta \mapsto \mathbb{E}_\theta(\hat{\theta} - \theta)^2$ is continuous. Then using the continuity and the properties above we can conclude the result.

1.1.1 Continuity of $\theta \mapsto \mathbb{E}_\theta(\hat{\theta} - \theta)^2$

I am going to handwave a proof of continuity of $\theta \mapsto \mathbb{E}_\theta(\hat{\theta} - \theta)^2$. We expand $\mathbb{E}_\theta(\hat{\theta} - \theta)^2$ as

$$c \int (\hat{\theta}(y) - \theta)^2 e^{-\frac{1}{2}(y-\theta)^2} \,\mathrm{d}\,y.$$

Then pick θ_0, θ_1 and consider

$$\frac{1}{c}\left(\mathbb{E}_{\theta_0}(\hat{\theta} - \theta_0)^2 - \mathbb{E}_{\theta_1}(\hat{\theta} - \theta_1)^2\right).$$

This can be rewritten into

$$\int (\hat{\theta}(y) - \theta_0)^2 e^{-\frac{1}{2}(y-\theta_0)^2} \,\mathrm{d}\,y - \int (\hat{\theta}(y) - \theta_1)^2 e^{-\frac{1}{2}(y-\theta_1)^2} \,\mathrm{d}\,y.$$

Using linearity of integrals, this becomes

$$\int (\hat{\theta}(y) - \theta_0)^2 e^{-\frac{1}{2}(y-\theta_0)^2} - (\hat{\theta}(y) - \theta_1)^2 e^{-\frac{1}{2}(y-\theta_1)^2} \,\mathrm{d}\,y.$$

Expanding the square and taking the same terms together, and splitting the integral we get that this is also equal to

$$\int \hat{\theta}(y)^2 \left(e^{-\frac{1}{2}(y-\theta_0)^2} - e^{-\frac{1}{2}(y-\theta_1)^2)}\right) \mathrm{d}\,y +$$
$$\int \hat{\theta}(y) \left(\theta_0 e^{-\frac{1}{2}(y-\theta_0)^2} - \theta_1 e^{-\frac{1}{2}(y-\theta_1)^2)}\right) \mathrm{d}\,y +$$
$$\int \theta_0^2 e^{-\frac{1}{2}(y-\theta_0)^2} - \theta_1^2 e^{-\frac{1}{2}(y-\theta_1)^2)} \,\mathrm{d}\,y$$

If θ_0 is close enough to θ_1, all these terms can be bounded by $\epsilon/3$, because we can bound the differences for fixed $\hat{\theta}$.

1.1.2 Bounding the integral

First note that $\mathbb{E}_\theta(Y-\theta)^2 = 1$, because $Y \sim N(\theta, 1)$, so $Y - \theta \sim N(0,1)$, and the second moment is variance plus mean squared, so $1 + 0^2$.
Now there exists a θ' such that $\mathbb{E}_{\theta'}(\hat{\theta}-\theta')^2 = 1-2\epsilon = \mathbb{E}_{\theta'}(Y-\theta')^2 - 2\epsilon$. So then there exists an open interval such that for all $\theta \in (\theta_0, \theta_1)$ we have $\mathbb{E}_\theta(\hat{\theta}-\theta)^2 \leq 1-\epsilon$.
So then we can estimate

$$\int \left(\mathbb{E}_\theta(Y-\theta)^2 - \mathbb{E}_\theta(\hat{\theta}-\theta)^2\right) \pi_\tau(\theta)\,\mathrm{d}\,\theta \geq \int_{\theta_0}^{\theta_1} \epsilon\pi_\tau(\theta)\,\mathrm{d}\,\theta,$$

by estimating the difference by 0 outside (θ_0, θ_1) and ϵ inside (θ_0, θ_1).
But the left hand side is just $1 - \int \mathbb{E}_\theta(\hat{\theta}-\theta)^2 \pi_\tau(\theta)\,\mathrm{d}\,\theta$, so we have shown what was needed.

1.2 Second part

Let $\tilde{\theta}_\tau$ be the posterior mean corresponding to the prior π_τ. Compute the corresponding Bayes risk

$$\int \mathbb{E}_\theta(\tilde{\theta}_\tau - \theta)^2 \pi_\tau(\theta)\,\mathrm{d}\,\theta.$$

1.2.1 posterior mean

We want to compute the posterior mean, that is

$$\int \theta\pi_\tau(\theta|y)\,\mathrm{d}\,\theta.$$

In order to do this quickly, we use that for the model $Y \sim N(\theta, 1)$, the prior $N(0, \tau)$ is conjugate. This means that the posterior is again in this family. The posterior is known in this case and is given by a normal random variable with mean $\frac{Y}{\frac{1}{\tau}+1}$ and variance $\frac{1}{\frac{1}{\tau}+1}$. So this gives the posterior mean, it is $\frac{Y\tau}{1+\tau}$.

1.2.2 Bayes risk

To compute the Bayes risk, we need to evaluate the following integral:

$$\int \mathbb{E}_\theta(\tilde{\theta}_\tau - \theta)^2 \pi_\tau(\theta)\,\mathrm{d}\,\theta.$$

As a first step we compute $\mathbb{E}_\theta(\tilde{\theta}-\theta)^2$. Since we known an expression of $\tilde{\theta}$ we can expand this into

$$\mathbb{E}_\theta(\frac{Y\tau}{1+\tau} - \theta)^2,$$

where

$$Y \sim N(\theta, 1).$$

Then

$$\frac{Y\tau}{1+\tau} \sim N\left(\frac{\theta\tau}{1+\tau}, (\frac{\tau}{1+\tau})^2\right).$$

So then we can compute the distribution of $\frac{Y\tau}{1+\tau} - \theta = \frac{Y\tau}{1+\tau} - \frac{\theta\tau+\theta}{1+\tau}$. This gives

$$\frac{Y\tau}{1+\tau} - \theta \sim N\left(\frac{-\theta}{1+\tau}, (\frac{\tau}{1+\tau})^2\right).$$

So this gives the mean squared error, which is variance plus mean squared:

$$\mathbb{E}_\theta(\tilde{\theta} - \theta)^2 = \frac{\theta^2}{(1+\tau)^2} + \frac{\tau^2}{(1+\tau)^2} = \frac{\theta^2 + \tau^2}{(1+\tau)^2}.$$

Then we plug this into the formula for the Bayes risk, this yields

$$\int \left(\frac{\theta^2}{(1+\tau)^2} + \frac{\tau^2}{(1+\tau)^2}\right) \pi_\tau(\theta) \,\mathrm{d}\,\theta.$$

Using linearity and that π_τ is a density we can simplify this into

$$\frac{1}{(1+\tau)^2} \int \theta^2 \pi_\tau(\theta) \,\mathrm{d}\,\theta + \frac{\tau^2}{(1+\tau)^2}.$$

We are again computing a second moment of a random variable, this time with mean zero and variance τ, so this yields

$$\frac{\tau}{(1+\tau)^2} + \frac{\tau^2}{(1+\tau)^2} = \frac{\tau^2 + \tau}{(1+\tau)^2} = \frac{\tau}{1+\tau}.$$

1.3 Third part

Using the previous results, show that is an estimator $\hat{\theta}$ as described would exist, then

$$\frac{1 - \int \mathbb{E}_\theta(\hat{\theta} - \theta)^2 \pi_\tau(\theta) \,\mathrm{d}\,\theta}{1 - \int \mathbb{E}_\theta(\tilde{\theta}_\tau - \theta)^2 \pi_\tau(\theta) \,\mathrm{d}\,\theta} \to \infty$$

as $\tau \to \infty$. Derive a contradiction.

As a first step we fill in what we know, this yields that we can bound this from below by

$$\frac{\epsilon \int_{\theta_0}^{\theta_1} \pi_\tau(\theta) \,\mathrm{d}\,\theta}{1 - \frac{\tau}{1+\tau}}$$

First we want to simplify this fraction a bit. This is after some algebra equal to

$$\epsilon \frac{1+\tau}{\sqrt{2\pi\tau}} \int_{\theta_0}^{\theta_1} e^{-\frac{1}{2\tau}\theta^2} \,\mathrm{d}\,\theta$$

We want to use the monotone convergence theorem. Observe that $e^{-\frac{1}{2\tau}\theta^2}$ converges to 1 for all θ as τ goes to infinity. Hence the integral converges, as τ goes to infinity to $\theta_1 - \theta_0$. So using the product rule for taking limits we get that the limit of the fraction is equal to the limit of

$$\epsilon(\theta_1 - \theta_0)\frac{1+\tau}{\sqrt{2\pi\tau}}$$

Which indeed diverges, as $\tau \to \infty$.

This means that there exists $\tau > 0$ such that

$$1 - \int \mathbb{E}_\theta(\hat{\theta} - \theta)^2 \pi_\tau(\theta) \,\mathrm{d}\,\theta \geq 1 - \int \mathbb{E}_\theta(\tilde{\theta}_\tau - \theta)^2 \pi_\tau(\theta) \,\mathrm{d}\,\theta$$

Which means that

$$\int \mathbb{E}_\theta(\hat{\theta} - \theta)^2 \pi_\tau(\theta) \,\mathrm{d}\,\theta \leq \int \mathbb{E}_\theta(\tilde{\theta}_\tau - \theta)^2 \pi_\tau(\theta) \,\mathrm{d}\,\theta$$

However, the posterior mean has minimal Bayes risk for the mean squared loss. So this yields a contradiction.