Formulation first part of the problem

For $m, n \in \mathbb{N}$ and sequences

$$\begin{aligned} &a_1, a_2, \dots, a_n \in \mathbb{R}, \\ &b_1, b_2, \dots, b_m \in \mathbb{R}, \\ &0 \le t_0 < t_1 < \dots < t_n < \infty, \\ &0 \le s_0 < s_1 < \dots < s_m < \infty, \end{aligned}$$

let $f(t) = \sum_{i=1}^n a_i \mathbb{1}_{(t_{i-1}, t_i]}(t)$, $g(t) = \sum_{i=1}^m b_i \mathbb{1}_{(s_{i-1}, s_i]}(\mathrm{t})$.
Define:

$$\mathbb{I}(f) = \sum_{i=1}^n a_i (W_{t_i} - W_{t_{i-1}}) \qquad \mathbb{I}(g) = \sum_{i=1}^n b_i (W_{s_i} - W_{s_{i-1}}).$$

Then we have to show:

$$\mathbb{E}\, \mathbb{I}(f)\mathbb{I}(g) = \int_0^\infty f(t)g(t)dt.$$

solution first part, part 1

We have:

$$\begin{aligned}
\mathbb{E}\,\mathbb{I}(f)\mathbb{I}(g) &= \mathbb{E}\sum_{i=1}^{n} a_i(W_{t_i} - W_{t_{i-1}})\sum_{j=1}^{m} b_j(W_{s_j} - W_{s_{j-1}}),\\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j \mathbb{E}(W_{t_i} - W_{t_{i-1}})(W_{s_j} - W_{s_{j-1}}),\\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j \mathbb{E}(W_{t_i}W_{s_j} - W_{t_i}W_{s_{j-1}} - W_{t_{i-1}}W_{s_j} + W_{t_{i-1}}W_{s_{j-1}}),\\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j \left((t_i \wedge s_j) - (t_i \wedge s_{j-1}) - (t_{i-1} \wedge s_j) + (t_{i-1} \wedge s_{j-1})\right).
\end{aligned}$$

solution first part, part 2

Now we analyze the summed term, for all i, j:

$$
\begin{aligned}
&(t_i \wedge s_j) - (t_i \wedge s_{j-1}) - (t_{i-1} \wedge s_j) + (t_{i-1} \wedge s_{j-1}) \\
&= \begin{cases}
t_i - t_i - t_{i-1} + t_{i-1} = 0 & \text{if } s_{j-1} > t_i, \\
s_j - s_j - s_{j-1} + s_{j-1} = 0 & \text{if } t_{i-1} > s_j, \\
 & \\
(t_i \wedge s_j) - s_{j-1} - t_{i-1} + (t_{i-1} \wedge s_{j-1}) & \text{if } (s_{j-1} \leq t_i) \wedge (t_{i-1} \leq s_j). \\
= (t_i \wedge s_j) - (t_{i-1} \vee s_{j-1}) &
\end{cases}
\end{aligned}
$$

As for all $x, y \in \mathbb{R}$:

$$
\begin{aligned}
(x \vee y) &= -(-x \wedge -y) = -((0 \wedge x - y) - x) \\
&= -((y \wedge x) - x - y) = x + y - (x \wedge y).
\end{aligned}
$$

solution first part, part 3

So actually

$$\mathbb{E}\, \mathbb{I}(f)\mathbb{I}(g) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j \left((t_i \wedge s_j) - (t_{i-1} \vee s_{j-1})\right) \mathbb{1}_{(s_{j-1}\le t_i)\wedge(t_{i-1}\le s_j)}.$$

Now we analyze the $L^2[0,\infty)$ inner product of f, g:

$$\int_0^\infty f(t)g(t)dt = \int_0^\infty \sum_{i=1}^{n} a_i \mathbb{1}_{(t_{i-1},t_i]}(t) \sum_{j=1}^{m} b_j \mathbb{1}_{(s_{j-1},s_j]}(t)dt$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j \int_0^\infty \mathbb{1}_{(t_{i-1},t_i]}(t)\mathbb{1}_{(s_{j-1},s_j]}(t)dt$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j \left((t_i \wedge s_j) - (t_{i-1} \vee s_{j-1})\right) \mathbb{1}_{(s_{j-1}\le t_i)\wedge(t_{i-1}\le s_j)} = \mathbb{E}\, \mathbb{I}(f)\mathbb{I}(g).$$

So indeed, for these arbitrary simple functions, we have $\mathbb{E}\, \mathbb{I}(f)\mathbb{I}(g) = \int_0^\infty f(t)g(t)dt$.

Solution second part of the problem, part 1

Show that these simple functions are dense in $L^2[0,\infty)$:
Consider Hermite functions

$$\psi_n(t) = (-1)^n (2^n n! \sqrt{\pi})^{\frac{1}{2}} e^{t^2/2} \frac{d^n}{dt^n} e^{-t^2}.$$

They form an orthonormal basis for $L^2(\mathbb{R})$ and are obviously continuous.

Let $f \in L^2[0,\infty)$, then $\exists (a_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{R})$ and functions $g_k := \sum_{n=1}^{k} a_n \psi_n$ such that $\|f \mathbb{1}_{[0,\infty)} - g_k\|_2 \overset{k\to\infty}{\to} 0$.
Hence g_k, restricted to $[0,\infty)$ is an approximation of f on $L^2[0,\infty)$.

Solution second part of the problem, part 2

Consider ψ_n^+ for some n, and the simple functions:

$$\psi_{n,1}^k = \sum_{i=0}^{k2^k-1} \mathbb{1}_{[i2^{-k},(i+1)2^{-k})} \inf_{t\in[i2^{-k},(i+1)2^{-k})} \psi_n(t)^+.$$

Then $\psi_{n,1}^k < \psi_n^+$ for all k and as ψ_n is continuous, ψ_n^+ is continuous and hence $\psi_{n,1}^k(t) \to \psi_n^+(t)$ for all $t \in \mathbb{R}$.
Now, $|\psi_n^+ - \psi_{n,1}^k|^2 \leq 4\,(\psi_n^+)^2$ and as $\lambda((\psi_n^+)^2) \leq \lambda(\psi_n^2) < \infty$, we have by dominated convergence that $\lambda(|\psi_n^+ - \psi_{n,1}^k|^2) \to 0$ as $k \to \infty$ and hence $\psi_{n,1}^k \stackrel{k\to\infty}{\to} \psi_n^+$ in $L^2[0,\infty)$.
By same reasoning, we have $\psi_{n,2}^k$ approximating ψ_n^- in $L^2[0,\infty)$.

Solution second part of the problem, part 3

We now see:

$$0 \leq \lambda((\psi_n - (\psi_{n,1}^k - \psi_{n,2}^k))^2) = \lambda(((\psi_n^+ - \psi_{n,1}^k) - (\psi_n^- - \psi_{n,2}^k))^2)$$
$$\leq 2(\lambda((\psi_n^+ - \psi_{n,1}^k)^2) + \lambda((\psi_n^- - \psi_{n,2}^k)^2)) \overset{k\to\infty}{\to} 0.$$

Hence ψ_n can be approximated by $\psi_n^k := \psi_{n,1}^k - \psi_{n,2}^k$.
Now

$$f = \lim_{r\to\infty} \sum_{n=1}^{r} a_n \psi_n = \lim_{r\to\infty} \lim_{k\to\infty} \sum_{n=1}^{r} a_n \psi_n^k. \qquad \text{(still simple)}.$$

Use bijection between $\mathbb{N}^2$ and $\mathbb{N}$ to see that for all $f \in L^2[0,\infty)$:

$$f(t) = \lim_{n\to\infty} \phi_n^f(t)$$

where ϕ_n^f is a sequence of simple functions restricted to intervals, depending on f. Hence these simple functions are dense in $L^2[0,\infty)$.

Solution third part of the problem

Show that $\int_0^\infty f(t)g(t)dt = \mathbb{E}\mathbb{I}(f)\mathbb{I}(g)$ **for all** $f, g \in L^2[0,\infty)$:
As $\mathbb{I}(\cdot)$ is an isometry we know that by the BLT-theorem, $\mathbb{I}$ can be extended to the whole of $L^2[0,\infty)$ with

$$\mathbb{I}(f) = \lim_{n\to\infty} \mathbb{I}(\phi_n^f) \in L^2(\mathbb{P}).$$

As strong convergence implies weak convergence in a Hilbert space, we have for all $f, g \in L^2[0,\infty)$ with g simple:

$$\int_0^\infty f(t)g(t)dt = \lim_{n\to\infty} \int_0^\infty \phi_n^f(t)g(t)dt = \lim_{n\to\infty} \mathbb{E}\mathbb{I}(\phi_n^f)\mathbb{I}(g) = \mathbb{E}\mathbb{I}(f)\mathbb{I}(g)$$

as both $L^2[0,\infty)$ and $L^2(\mathbb{P})$ are Hilbert spaces.
Repeat the procedure to show the identity for arbitrary f, g.

That's it..

Questions?

extra stuff

Consider

$$\int_{\mathbb{R}} (f \mathbb{1}_{[0,\infty)} - g_k)^2 dt = \int_{\mathbb{R}} ((f - g_k)\mathbb{1}_{[0,\infty)} - g_k \mathbb{1}_{(-\infty,0)})^2 dt$$

$$= \int_0^\infty (f - g_k)^2 dt + \int_{-\infty}^0 g_k^2 dt \overset{k\to\infty}{\to} 0.$$

Hence g_k restricted to $[0,\infty)$ also approximation of f in $L^2[0,\infty)$.