Minimax risk over large classes of signals
Statistical theory for high- and infinite-dimensional models

Jolien Oomens
May 10th, 2017
Some $L^2$ theory

Recall we write

$$e_n = e^{2\pi i n x},$$

and that

$$\{e_n : n \in \mathbb{Z}\}$$

is an orthonormal basis of $L^2[0,1]$.

The distance between $e_n$ and $e_m$ ($m \neq n$) is

$$\|e_n - e_m\|_2^2 = \langle e_n - e_m, e_n - e_m \rangle = 1 + 1 = 2.$$
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Exercise 5.1(i)

Prove that for every $p > 0$ we have

$$\inf \sup E_f \| \hat{f} - f \|_{L^2}^p \rightarrow \infty \text{ as } n \rightarrow \infty.$$
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Letting $n \to \infty$ now proves the desired equality.
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Prove that for $p > 0$ there exists $c > 0$ such that

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Letting $n \to \infty$ we obtain desired equality.
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Let $\mathcal{F} \subset L^2[0, 1]$ be bounded but not totally bounded. Show that for $p > 0$ there is a $c > 0$ such that $\inf_{\hat{f}} \sup_{f \in \mathcal{F}} E_f \|\hat{f} - f\|_{L^2}^p \geq c$. 

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$$e^{\frac{1}{2} n \text{diam}(\mathcal{G}_n)^2} = e^{2nM^2} \leq |\mathcal{G}_n| = e^{n^2},$$
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$$\frac{1}{2} e^{n} \text{diam}(\mathcal{G}_n)^2 = e^{2nM^2} \leq |\mathcal{G}_n| = e^{n^2},$$

so the proposition gives the result.