Minimax risk over large classes of signals

Statistical theory for high- and infinite-dimensional models

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Prove that for every $p > 0$ we have

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Letting $n \to \infty$ now proves the desired equality.

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