Smoothness and decay of Fourier coefficients

Statistical theory for high- and infinite-dimensional models

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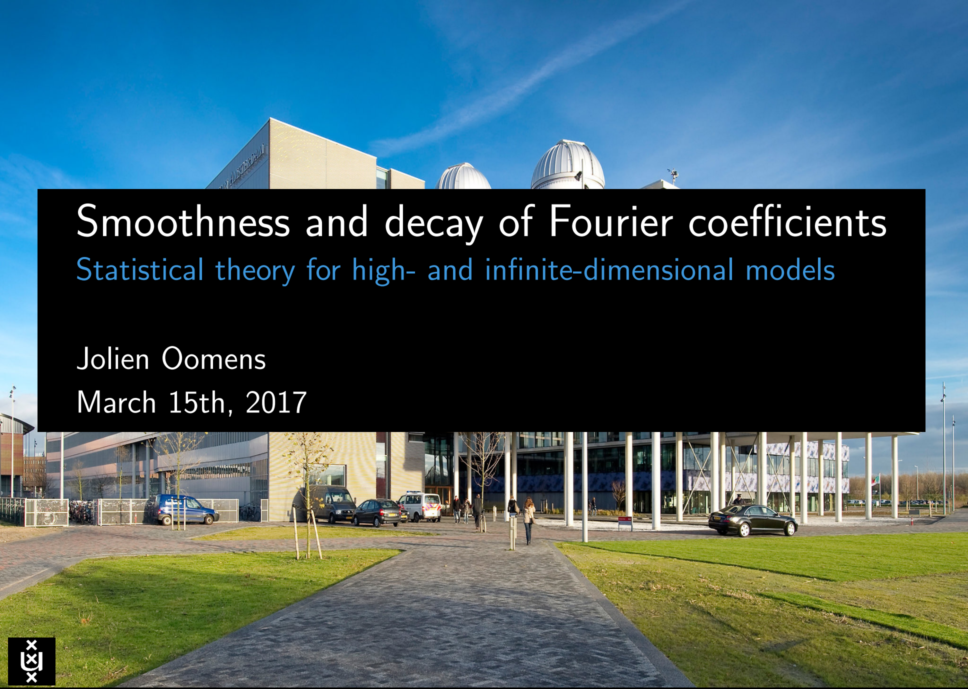

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Since we identify the endpoints 0 and 1, we can repeat this argument for higher order derivatives.

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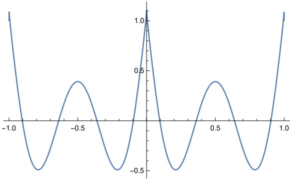

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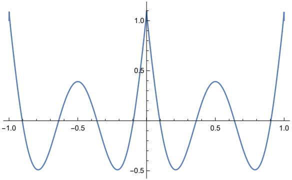

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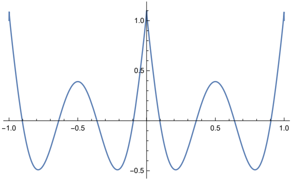

How is "differentiability" defined for L^2-functions? We say g is the derivative of f iff $f(x) - f(0) = \int_0^x g$ for almost every x.

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The Fourier coefficients of $f^{(\beta)}$ are $f_k^{(\beta)} = (2\pi i k)^\beta f_k$.

Theorem: Plancherel

For any $f \in L^2[0,1]$ we have $\sum |f_j|^2 = \|f\|_{L^2}^2$.

Suppose that f is chosen such that $f^{(\beta)} \in L^2[0,1]$. We have

$$\left\|f^{(\beta)}\right\|_{L^2}^2 = \sum \left|f_k^{(\beta)}\right|^2 = \sum (2\pi k)^{2\beta} |f_k|^2 =$$

Exercise

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Since $\|f^{(\beta)}\|_{L^2}^2 < \infty$ we have $\sum k^{2\beta}|f_k|^2 < \infty$ as well. □