4.1 Complete the proof of Proposition 4.2.1.

Solution. We need to show that

$$\inf_{\alpha\in(0,1)} \left(\alpha + 1 - \Phi\left(\sqrt{n}\left\|f_1 - f_0\right\|_2 - \xi_{1-\alpha}\right)\right) = 1 - \Phi\left(\frac{1}{2}\sqrt{n}\left\|f_1 - f_0\right\|_2\right)$$

Because the expression on the RHS we are taking the infimum over is differentiable we can find a minimum by solving the first order conditions (setting the derivative to zero). Since the infimum is a value that is attained it coincides with the minimum.

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\alpha^* + 1 - \Phi\left(\sqrt{n}\left\|f_1 - f_0\right\|_2 - \Phi^{-1}(1-\alpha^*)\right)\right) = 0$$

The chain rule gives us:

$$-\Phi'\left(\sqrt{n}\left\|f_1 - f_0\right\|_2 - \Phi^{-1}(1-\alpha^*)\right) * -\left(\Phi^{-1}\right)'(1-\alpha^*) * -1 = -1$$

By the Inverse Function theorem:

$$\frac{\Phi'(\sqrt{n}\left\|f_1 - f_0\right\|_2 - \Phi^{-1}(1-\alpha^*))}{\Phi'\left(\Phi^{-1}(1-\alpha^*)\right)} = 1$$

By the Fundamental theorem of Calculus

$$\frac{\varphi(\sqrt{n}\left\|f_1 - f_0\right\|_2 - \Phi^{-1}(1-\alpha^*))}{\varphi\left(\Phi^{-1}(1-\alpha^*)\right)} = 1$$

where φ is the pdf of a standard Normal rv. Now we can easily solve for the minimum:

$$\begin{aligned}
&\exp\left[-\frac{1}{2}\left[\left(\sqrt{n}\left\|f_1 - f_0\right\|_2 - \Phi^{-1}(1-\alpha^*)\right)^2 - \left(\Phi^{-1}(1-\alpha^*)\right)^2\right]\right] = 1\\
\Leftrightarrow &\exp\left[-\frac{1}{2}\left[\left(\sqrt{n}\left\|f_1 - f_0\right\|_2\right)^2 - 2\sqrt{n}\left\|f_1 - f_0\right\|_2\Phi^{-1}(1-\alpha^*)\right]\right] = 1\\
\Leftrightarrow &\Phi^{-1}(1-\alpha^*) = \frac{1}{2}\sqrt{n}\left\|f_1 - f_0\right\|_2
\end{aligned}$$

Thus it indeed follows that

$$\alpha^* = 1 - \Phi\left(\frac{1}{2}\sqrt{n}\left\|f_1 - f_0\right\|_2\right)$$

It can analogously be shown that the second order derivative is positive thus we indeed have a minimum.

□

4.2 Proof the claims about the C^β- and L^2-norms of the smooth bump function f_σ in Example 4.2.3

Solution.

$$f_\sigma(x) = \sigma^\beta K\left(\frac{x-1/2}{\sigma}\right) \quad , x \in (0,1) \text{ and } K \text{ is } C^\infty \text{ and has compact support.}$$

$$\begin{aligned}
\|f_\sigma\|_2^2 &= \int_{(0,1)} f_\sigma(x)^2 \, \mathrm{d}\lambda = \int_0^1 \left[\sigma^\beta K\left(\frac{x-1/2}{\sigma}\right)\right]^2 \mathrm{d}x \\
&= \sigma^{2\beta} \int_0^1 \left[K\left(\frac{x-1/2}{\sigma}\right)\right]^2 \mathrm{d}x \\
&= \sigma^{2\beta+1} \int_{-1/2\sigma}^{1/2\sigma} K^2(u) \, \mathrm{d}u \quad \text{, substitution } u = \frac{x-1/2}{\sigma}
\end{aligned}$$

Due to the compactness of support of K the integral is finite thus indeed the smooth spike is consistently detectable if and only if $n\sigma^{2\beta+1}$ tends to infinity.

$$\|f_\sigma\|_{C^\beta} = \max_{k \le \underline{\beta}} \left\|f_\sigma^{(k)}\right\|_\infty + \sup_{s \neq t} \frac{|f_\sigma^{\underline{\beta}}(t) - f_\sigma^{\underline{\beta}}(s)|}{|t-s|^{(\beta-\underline{\beta})}}$$

In order to bound the norm we look at both terms separately. Not that for the k-th derivative of the bump function ($k \le \underline{\beta}$) it holds that:

$$f_\sigma^{(k)} = \sigma^{(\beta-k)} K^{(k)}\left(\frac{x-1/2}{\sigma}\right)$$

All derivatives of K are continuous and defined on a compact set thus have a global maximum and can thus be bounded. By assumption σ is close to zero and now it follows that the first term of the Holder norm is bounded.

For the second term in the Holder norm we get the following:

$$\sup_{s \neq t} \frac{|f_\sigma^{\underline{\beta}}(t) - f_\sigma^{\underline{\beta}}(s)|}{|t-s|^{(\beta-\underline{\beta})}} = \sup_{s \neq t} \frac{\left|\sigma^{\beta-\underline{\beta}}\left[K^{\underline{\beta}}\left(\frac{t-\frac{1}{2}}{\sigma}\right) - K^{\underline{\beta}}\left(\frac{s-\frac{1}{2}}{\sigma}\right)\right]\right|}{|t-s|^{(\beta-\underline{\beta})}}$$

Because K is C^∞ it is also Holder continuos thus for some constant C:

$$\left|K^{\underline{\beta}}\left(\frac{t-\frac{1}{2}}{\sigma}\right) - K^{\underline{\beta}}\left(\frac{s-\frac{1}{2}}{\sigma}\right)\right| \le C \frac{|t-s|^{(\beta-\underline{\beta})}}{\sigma^{\beta-\underline{\beta}}}$$

Thus it follows that the second term in the Holder norm is bounded by C. Thus we see that both components of this norm are bounded and $\|f_\sigma\|_{C^\beta}$ is therefore bounded. □