Statistical Theory for High- and Infinite-dimensional Models

Ardjen Pengel

Exercise 2.0+
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Theorem: Ito isometry

Let $f \in L^2[0, \infty)$ and W_t be a standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$ then :

$$\mathbb{E}[I(f)^2] = \int f(t)^2 dt$$

Consequently,

$$\mathbb{E}[I(f)I(g)] = \int f(t)g(t)dt$$

Proof

Suppose first that f is simple i.e. $f = \sum_{j=0}^{n} a_j 1_{(t_j, t_{j+1}]}$,where $0 = t_0 < \cdots < t_n = t$ and $a_j \in \mathbb{R}$.

$$I(f) = \sum_{j=0}^{n} a_j (W_{t_{j+1}} - W_{t_j}) = \sum_{j=0}^{n} a_j D_j$$

where $\mathrm{D}_j = W_{t_{j+1}} - W_{t_j}$.

$$I(f)^2 = \sum_{j=0}^{n} {a_j}^2 D_j^2 + 2 \sum_{0 \le i < j \le n} a_i a_j D_i D_j$$

Lets first evaluate the expectation of the cross terms:

$$\mathbb{E}[a_i a_j D_i D_j] = \mathbb{E}[a_i a_j \mathbb{E}[D_i D_j | \mathcal{F}(t_j)]]$$

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Note that D_i is $\mathcal{F}(t_j)$ measurable and D_j is independent of $\mathcal{F}(t_j)$

$$\mathbb{E}[a_i a_j \mathbb{E}[D_i D_j | \mathcal{F}(t_j)]] = \mathbb{E}[a_i a_j \mathbb{E}[D_j]] = 0$$

Thus we see that

$$\mathbb{E}[I(f)^2] = \sum_{j=0}^{n} {a_j}^2 \mathbb{E}[D_j^2] = \sum_{j=0}^{n} {a_j}^2 (t_{j+1} - t_j)$$

$$= \int_0^\infty \sum_{j=0}^{n} {a_j}^2 1_{(t_j, t_{j+1}]} dt = \int_0^\infty f(t)^2 dt$$

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Proof

We have shown that $I : S \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ preservers norms and thus is a linear isometry.

$$\|I(f)\|_{L^2(\Omega)} = \left[\mathbb{E}[I(f)^2]\right]^{\frac{1}{2}} = \left[\int f(t)^2 dt\right]^{\frac{1}{2}} = \|f\|_{L^2[0,\infty)}$$

Any linear isometric mapping also preserves the inner product:

$$\forall f, g \in S : \langle I(f), I(g) \rangle_{L^2(\Omega)} = \langle f, g \rangle_{L^2[0,\infty)}$$

This gives the required result:

$$\mathbb{E}[I(f)I(g)] = \int f(t)g(t)dt$$

This can be extended to general functions f and g by the "Cauchy sequence an completeness" argument and an MCT for interchanging the norm and the limit.

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Theorem

The simple functions S are dense in $L^2[0,\infty)$. Every measurable f is the pointwise limit of simple funtions.

$$\exists \{f_n\}_{n=1}^{\infty} \subset S \quad \text{s.t.} \quad f = \sup_n f_n$$

Proof: The idea is to divide the range of f.
First divide $[0, 2^n]$ into 2^{2n} subintervals of width 2^{-n}.
Define for every n:

$$A_{k,n} = f^{-1}[(k2^{-n}, (k+1)2^{-n})]], \quad k = 0, 1, \cdots 2^{2n} - 1)$$

and $B_n = f^{-1}[(2^n, \infty)]$.
This gives us the approximating simple functions:

$$f_n(x) := \sum_{k=0}^{2^{2n}-1} k2^{-n} 1_{A_{k,n}} + 2^n 1_{B_n}$$

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Note that indeed
(i) $A_{k,n}$ and B_n are in $\mathcal{F}$ by the measurability of f
(ii) $0 \leq f_n \leq f$ and $f_n \uparrow f$

These simple funtions are elements of $L^2[0,\infty)$ and thus $\forall f \in L^2[0,\infty)$ there is a sequence of simple functions $\{f_n\}$ s.t. by the DCT

$$\lim_{n\to\infty} \int |f - f_n|^2 dt = 0$$

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It must be noted that the actual excercise was proving that the set $\{f : f = \sum_{j=1}^{n} a_j 1_{(t_j, t_{j+1}]}\}$ was dense in $L^2[0, \infty)$.

This can be done by using the fact that the set of compactly supported continuous functions are dense in $L^2[0, \infty)$. With similar argumentation as the preceding proof it is enough to approximate the L^2 functions on a compact interval because by the integrability assumption they must be bounded almost everywhere. And using lower Riemann sums to approximate the continuous function from below gives us the required result.