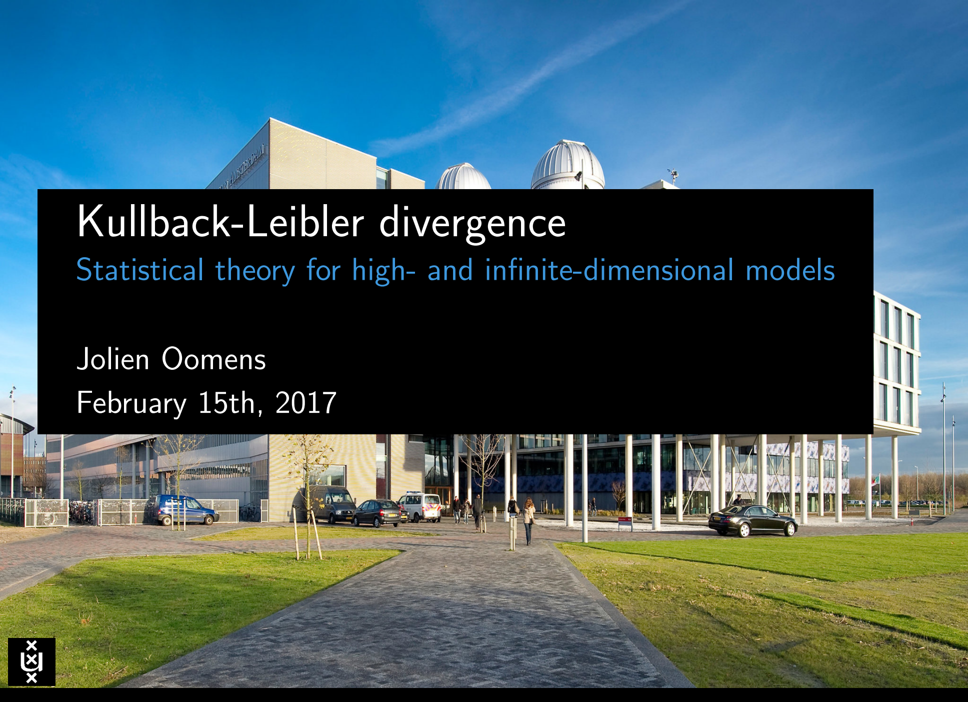

Kullback-Leibler divergence

Statistical theory for high- and infinite-dimensional models

Jolien Oomens

February 15th, 2017

Kullback-Leibler divergence

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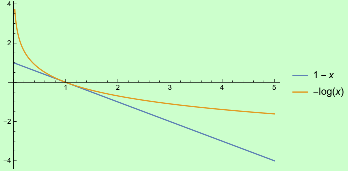

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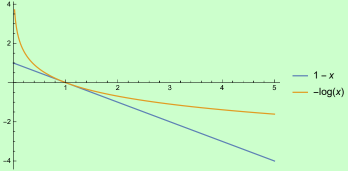

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